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Investigating the law of tendential fall in the rate of profit based on feedback control

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ABSTRACT

The rate of profit is a key element for understanding the movement of capitalism such as technological progress and economic crisis. Even though capitalists seek larger profit rates, there exists a tendency of stagnating or falling profit rates. According to Marx, the rate of profit would tend to decline in the long run as a result of technological progress, termed the law of tendential fall in the profit rate. This article introduces a novel approach based on the discrete-time feedback control mechanism to elucidate Marx's theory. Specifically, this study presents a mechanism and conditions to show how the effort to maximize the profit rate, implemented by designing an "appropriate" control law under the sampling of fiscal years, eventually leads to the gradual decrease in the rate of profit in the long run.

1. Introduction

Not just do capitalists try to obtain more profit, they also pursue a steadily increasing profit rate. However, it seems that the rate of profit would not easily rise in many rich capitalist countries. For instance, the profit rate in the United States (U.S.) economy has a declining trend for the long period of 1945–2015 as shown in Fig. 1 (computed using data in Duménil and Lévy (2016)). Marx developed a controversial theory for the rate of profit that it would tend to decline in the long term (Marx, 1991). According to Marx, $r(t)$, the rate of profit at a fiscal year $t \in \mathbb{N}$, is

$$r(t) = \frac{\alpha(t)v(t)}{c(t) + v(t)} = \frac{\alpha(t)}{c(t)/v(t) + 1}, \tag{1}$$

where *constant capital* $c(t)$ is the value of means of production such as materials, machinery, and factory, and *variable capital* $v(t)$ is the wage paid to workers. Representing the intensity of exploitation, the *surplus-value rate* $\alpha(t)$ increases as does the intensity of labor or the length of working day while the wage remains constant. $c(t)/v(t)$ is called the *composition* (or value composition) of capital. For example, for a capitalist who invests 100 dollars and earns 115 dollars, if he invests 60 dollars for materials and machinery and pays 40 dollars as the wage for a worker, then the constant capital and variable capital are 60 and 40 dollars, respectively, and the composition of capital is $60/40 = 1.5$. Since the profit is 15 dollars, the surplus-value rate is $15/40 = 0.375$.

In Marx's theory, technological progress during a long period of expansion raises the composition of capital and, as a result, the rate of profit would decline in the long run. The rise in the composition of capital means that one worker operates with constant capital of ever-growing scale, and consequently the productivity of labor increases. However, the fall in the profit rate will eventually be followed by a severe depression, during which the rate of profit would tend to increase primarily due to the devaluation of capital. As a corroborating case, the rate of profit in the U.S. increased from 0.02 in the Great Depression in 1932 to 0.31 in 1944.

Many research attempts are found on Marx's theory of the falling rate of profit. Two major issues exist regarding this subject, namely, whether the profit rate really falls in the long run, and if so, what is the main reason—the fall in the surplus-value rate or the rise in the composition of capital? As per the post-war U.S. economy, for instance, Weisskopf (1979) argues that the rate of profit declined because of a fall in the surplus-value rate, which is supported by several economists including Bakir and Campbell (2006). Moseley (1991) contends that since the composition of capital increased faster than the rate of surplus-value, the decline of the profit rate was inevitable. Jones (2016) also specifies the

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Nomenclature

List of notations

$r(t)$	Rate of profit at year t
$c(t)$	Constant capital at year t
$v(t)$	Variable capital at year t
$\alpha(t)$	Surplus-value rate at year t
$p(t)$	Profit at year t
β	Accumulation rate
$\gamma(t)$	Accumulation composition at year t
$l(t)$	Number of workers at year t
$w(t)$	Average wage per worker at year t
$\rho(x(t))$	Growth rate of a variable $x(t)$
$y(t)$	Output at year t
$a(t)$	State of technology at year t
$\pi(t)$	Profit share at year t

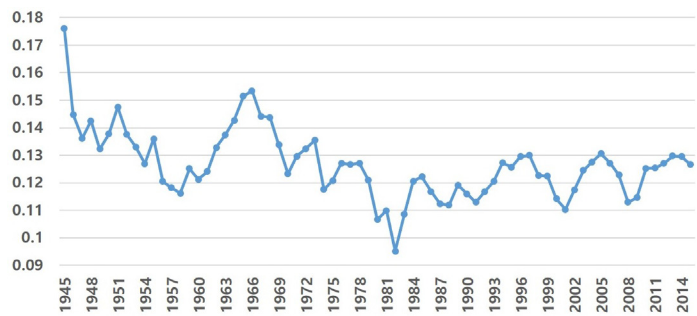


Fig. 1. Profit rate in the U.S. for 1945–2015.

main reason of the fall as the rise in the composition of capital. According to Petith (2005), on the other hand, Marx was never able to provide a demonstration that the rate of profit must fall and he was aware of this.

In model-based approaches, Duménil and Lévy (2003) present a dynamic model with no control mechanisms to describe the tendential fall in the profit rate. They interpret the law of tendential fall as a statement concerning the specific feature of innovation adopted to yield a larger profit rate. In the authors' prior work (Park and Yang, 2023), on the other hand, a dynamic model having a feedback control law is presented. The control input is intended to increase the profit rate in the next year, but it provokes further increases in both composition of capital and surplus-value rate, which leads to the inevitable fall in the profit rate.

Based on the dynamic model of the profit rate in Park and Yang (2023), this article presents a novel control law to maximize both the profit and its rate in the next year. To this end, we refine the state space model of Park and Yang (2023) so that it can accurately accommodate the variable perturbations stemming from the invoked control input. We show that despite the use of the control policy for profit maximization, the decline in the profit rate is unavoidable. We also elaborate on certain conditions for the tendential fall in the short and long runs. Further, numerical experiments are conducted by applying the proposed methodology to real data of the U.S. economy from 1945 to 2015 and that of the United Kingdom (U.K.) economy from 1950 to 2019.

Various studies are found on profit maximization in the field of information technology, such as cloud computing centers (Liu et al., 2021), crowdsensing systems (Liu et al., 2020), electric vehicles (Yang et al., 2014), airline ticket prediction (Abdella et al., 2021), and stakeholders of crossover services (Liu et al., 2022). There also exist a lot of research attempts that maximize the profit in portfolio selection problems using optimization theory; refer to Khan et al. (2022) and Wang et al. (2023) for recent results. On the other hand, control theoretic approaches to solving economic problems are rarely found, e.g., profit maximization in duopoly (Kogan et al., 2016), profit maximization and advertising (Leitmann and Schmitendorf, 1978), stock trading (Hsieh, 2022), fashion industry (Kort et al., 2006), etc.

In comparison with the prior work, the present study has the following contributions and advantages.

1) Marx's theory asserts that the effort of increasing the rate of profit eventually leads to its progressive fall, which is an inherent contradiction of the capitalist production mode. Among the cited publications (Bakir and Campbell, 2006; Duménil and Lévy, 2003; Jones, 2016; Moseley, 1991; Petith, 2005; Weisskopf, 1979), however, no prior work exists which stipulates any technical conditions to explain such a contradiction. By contrast, this study reveals the relations between the parameters of a production function, a wage model, and the state of technology yielding the contradictory feature in the framework of the state space.
2) There exists no literature which studies the relationship between the composition of capital and surplus-value rate using dynamic models, though it is crucial for understanding Marx's theory. Especially, most of the classical approaches regard the surplus-value rate as constant; see, e.g., Foley et al. (2019). In this article, however, the surplus-value rate is considered a variable. It will be shown that the growth of the surplus-value rate is proportional to that of composition of capital.
3) All of the prior work on profit maximization (Abdella et al., 2021; Khan et al., 2022; Kogan et al., 2016; Leitmann and Schmitendorf, 1978; Liu et al., 2021; Liu et al., 2020; Liu et al., 2022; Wang et al., 2023; Yang et al., 2014) lacks the results of analyzing the dynamic behavior of the profit rate in a nation's economy using feedback control as done in this study. Further, the proposed method ensures practicality as it is verified by the numerical experiments with respect to real U.S. and U.K. economy data.

Finally, we note that the present study differs from the authors' prior work (Park and Yang, 2023) in that while Park and Yang

(2023) employs an arbitrary control input increasing the profit rate, the present study designs a meticulous control law aiming at the maximization of both the profit and its rate. The associated state space model is also amended to the extent that the behavior of the profit rate is accurately described with an infinitesimal value of the control input.

2. Main results

2.1. Dynamic model of profit rate

A basic premise of Marx's theory is that the value of a commodity, represented by a price, is determined by the quantity of needed labour, which is in turn measured by the socially necessary labour-time to produce the commodity. The profit is a part of the value created by workers' labour called surplus-value. From this theory of value, $p(t)$, the total profit at year t, is described by the product of surplus-value rate and variable capital as follows:

$$p(t) = \alpha(t)v(t). \tag{2}$$

It can be said that the rate of profit $r(t)$ in (1) is alternatively defined as the ratio of $p(t)$ to total capital $c(t) + v(t)$.

In view of (1), two factors determine the profit rate $r(t)$—the surplus-value rate $\alpha(t)$ and the composition of capital $c(t)/v(t)$. Obviously, $r(t)$ decreases as $c(t)/v(t)$ increases when $\alpha(t)$ is constant. In reality, however, the surplus-value rate fluctuates, making it difficult to determine whether or not the rate of profit decreases in accordance with the increase in composition of capital. This is the main reason for the controversy surrounding Marx's law of the tendential fall in the rate of profit. Marx states that as the composition of capital increases, the surplus-value rate also tends to increase (Marx, 1991).

A portion of the profit $p(t)$ is accumulated as capital in the next year $t+1$. Denote the accumulated capital by $\beta p(t)$ where the *accumulation rate* β $(0 \leqslant \beta \leqslant 1)$ is assumed to be constant. $\beta p(t)$ is further divided into the accumulated constant capital $\Delta c(t)$ and variable capital $\Delta v(t)$, that is,

$$\beta p(t) = \Delta c(t) + \Delta v(t). \tag{3}$$

Thus the variable capital and constant capital at year $t+1$ are

$$\begin{aligned} v(t+1) &= v(t) + \Delta v(t), \\ c(t+1) &= c(t) + \Delta c(t). \end{aligned}$$

As the control input, we introduce $\gamma(t)$ called the *accumulation composition* of capital which is defined as

$$\gamma(t) = \frac{\Delta c(t)}{\Delta v(t)}. \tag{4}$$

We assume that $\gamma(t) \geqslant 0$, and hence the constant and variable capitals always increase except for the case of $\gamma(t) = 0$. However, in a severe depression like the Great Depression in 1932, this assumption may not hold. The constant capital may decrease due to depreciation of capital goods and the variable capital may shrink owing to an increase of unemployment. Hence the assumption of $\gamma(t) \geqslant 0$ stipulates that the present study considers only the period of expansion, which conforms to the Marx's statement that the rate of profit tends to decline during a long period of expansion. As the definition (4) implies, $\Delta v(t)$ is supposed to be always nonzero. If $\Delta v(t)$ is close to zero, $\gamma(t)$ will approach to ∞, which means that almost all of accumulated capital $\beta p(t)$ will be invested in constant capital, e.g., buying machines and building factories.

From (2)–(4), we have

$$\Delta v(t) = \frac{\beta\alpha(t)}{1+\gamma(t)} v(t). \tag{5}$$

Let $l(t)$ and $w(t)$ be the number of workers and the average wage per worker at year t, respectively. Since the variable capital equals the total wage paid to workers, it holds that

$$v(t) = w(t)l(t). \tag{6}$$

Then, the variable capital at year $t+1$ is

$$\begin{aligned} v(t+1) &= w(t+1)l(t+1) \\ &= (w(t) + \Delta w(t))(l(t) + \Delta l(t)) \\ &= w(t)l(t) + w(t)\Delta l(t) + l(t)\Delta w(t) + \Delta w(t)\Delta l(t) \\ &\simeq v(t) + w(t)\Delta l(t) + l(t)\Delta w(t). \end{aligned} \tag{7}$$

In most cases, the last approximation holds true since $\Delta w(t)\Delta l(t)$ is very small compared to $w(t)\Delta l(t) + l(t)\Delta w(t)$. For instance, in the U. S. economy during 1946–2015, the average ratio of $\Delta w(t)\Delta l(t)$ to $w(t)\Delta l(t) + l(t)\Delta w(t)$ was merely 0.017. The derivation (7) in turn implies

$$\Delta v(t) \simeq w(t)\Delta l(t) + l(t)\Delta w(t). \tag{8}$$

It should be noted that the approximation (7) is not valid when the control input $\gamma(t)$ is close to zero. In the case of $\gamma(t) = 0, \Delta c(t) = 0$ and $\Delta v(t) = \beta p(t)$, i.e., all the accumulated capital is invested as variable capital. This means that $\Delta w(t)\Delta l(t)$ should not be ignored. Nevertheless, the approximation is deemed valid for typical values of $\gamma(t)$ which increase the profit and its rate in the next year.

Let $\rho(x(t))$ denote the growth rate of a variable $x(t)$, i.e.,

$$\rho(x(t)) = \frac{\Delta x(t)}{x(t)} = \frac{x(t+1) - x(t)}{x(t)}.$$

If both $\rho(x)$ and $\rho(z)$ are small for two variables x and z, then $\rho(x/z) \simeq \rho(x) - \rho(z)$ and $\rho(xz) \simeq \rho(x) + \rho(z)$. As a wage model, we adopt the following equation presented by Duménil and Lévy (2003).

$$\rho(w(t)) = \delta\rho(l(t)) + \lambda, \tag{9}$$

where δ and λ are typically positive parameters. This model states that $\rho(w(t))$, the growth rate of wage, is proportional to $\rho(l(t))$, that of employment. δ captures the effect of the employment increase on the wage, and also describes the bargaining power of workers. λ explains the phenomenon that even though employment decreases, wage may increase.

Among various wage models in traditional growth theories, the formulation (9) is perceived to well account for Marx's analytical framework. According to Marx, the general movement of wages, or the demand and supply of labor, is exclusively regulated by the expansion and contraction of capital since "the labor-market sometimes appearing relatively under-supplied because capital is expanding, and sometimes relatively over-supplied because it is contracting." (Marx, 1991).

Let $y(t)$ be an output at year t representing the total value created by workers' labor at the year. $y(t)$ is divided into the total wage $v(t)$ and profit $p(t)$, namely,

$$y(t) = p(t) + v(t). \tag{10}$$

While many production functions with respect to the output are available in economics, here we adopt the following Cobb-Douglas production function

$$y(t) = a(t)c^m(t)l^{1-m}(t), \tag{11}$$

where $0 < m < 1$ and $a(t)$ is the *state of technology* implying that a larger $a(t)$ produces a larger output with a given number of workers and constant capital. Since $a(t)$ is a quality factor as opposed to quantitative $c(t)$ and $l(t)$, it has no unit. We assume that the state of technology is exogenously given by

$a(t) = a_0 e^{\epsilon t} (\epsilon > 0)$.

Then the rate of technological progress is reduced to

$\rho(a(t)) = \epsilon$.

In the mainstream economics (called neoclassical economics), m in (11) is set to be the profit share in order to justify the most important neoclassical hypothesis that the marginal product of labor is equal to a real wage, which guarantees the profit maximization. Specifically, from (6) and (10) (remind that $l(t)$ is the number of workers at year t),

$$\begin{aligned}\frac{\partial p}{\partial l} &= \frac{\partial y}{\partial l} - \frac{\partial v}{\partial l}\\ &= (1-m)a(t)c^m(t)l^{-m}(t) - w(t)\\ &= (1-m)\frac{y(t)}{l(t)} - w(t).\end{aligned}$$

If m is the profit share, i.e., $m = p(t)/y(t)$, the above equation is reduced to

$$\begin{aligned}\frac{\partial p}{\partial l} &= \left(1 - \frac{p(t)}{y(t)}\right)\frac{y(t)}{l(t)} - w(t)\\ &= \frac{v(t)}{y(t)}\frac{y(t)}{l(t)} - w(t) \quad (\Leftarrow (10))\\ &= 0. \qquad\qquad\qquad\quad (\Leftarrow (6))\end{aligned}$$

That is, $\partial y/\partial l$, the marginal product of labor, equals $w(t)$.

However, there are some critics of the foregoing neoclassical hypothesis. For example, Foley et al. (2019) argue that in reality the average wage is greater than the marginal product of labor. They criticize the smooth production functions such as the Cobb-Douglas function wherein the profit rate maximizing technique at a given wage will always combine labor and capital in proportions such that the marginal product of labor is equal to the wage. This criticism is valid only if m is equal to the profit share. With m differing from a profit share, the latter assertion loses credibility. To discuss this point in detail, assume that m is not equal to the profit share by letting

$m = \pi(t) + \pi'$,

where $\pi(t) = p(t)/y(t)$ is the profit share at year t. With this m, the marginal product of labor is derived as

$$\begin{aligned}\frac{\partial y}{\partial l} &= (1-m)\frac{y(t)}{l(t)}\\ &= (1-\pi(t))\frac{y(t)}{l(t)} - \pi'\frac{y(t)}{l(t)}\\ &= w(t) - \pi'\frac{y(t)}{l(t)}.\end{aligned}$$

One can see from the above result that insofar as $\pi' \neq 0$, the marginal product of labor $\partial y/\partial l$ can be never equal to the wage $w(t)$. In particular, If $m > \pi(t)$ $(\pi' > 0)$, $w(t) > \partial y/\partial l$ and if $m < \pi(t)$ $(\pi' < 0)$, $w(t) < \partial y/\partial l$. This analysis shows that contrary to the claim that the wage is greater than the marginal product of labor in reality (Foley et al., 2019), the reversed outcome may occur.

Although the Cobb-Douglas function is criticized by the classical economists, it inherits Marx's purview that the increase in the composition of capital is another expression for the development of the social productivity of labor, which is shown by the way that the growing use of machinery and fixed capital generally enables more raw and ancillary materials to be transformed into products in the same time by the same number of workers, i.e., with less labor. Another form of the Cobb-Douglas function is $y(t)/l(t) = a(t)(c(t)/l(t))^m$ with the implication that if $c(t)/l(t)$, the fixed capital transformed into products by one worker, increases, so does the productivity of labor $y(t)/l(t)$.

We now present a dynamic system model having a feedback control mechanism with state variables $l(t), w(t), c(t)$, output variables $y(t), \alpha(t), r(t)$, and the control input $\gamma(t)$. From (2), (6), and (10), we obtain the surplus-value rate as follows.

$$\begin{aligned}\alpha(t) &= \frac{y(t) - w(t)l(t)}{w(t)l(t)}\\ &= \frac{a(t)c^m(t)l^{1-m}(t) - w(t)l(t)}{w(t)l(t)}.\end{aligned}$$

From (8) and (9), in addition, we compute

$$\Delta l(t) = \frac{1}{1+\delta}\left(\frac{\beta\alpha(t)}{1+\gamma(t)} - \lambda\right) l(t). \tag{12}$$

Finally, deriving $\Delta w(t)$ and $\Delta c(t)$ by referring to the preceding equations, we construct the following nonlinear dynamic model (Park and Yang, 2023).

State equations:

$$\begin{aligned}&l(t+1) = l(t) + \Delta l(t),\\ &\Delta l(t) = \frac{1}{1+\delta}\left(\frac{\beta}{1+\gamma(t)}\frac{a(t)c^m(t)l^{1-m}(t) - w(t)l(t)}{w(t)l(t)} - \lambda\right) l(t),\\ &w(t+1) = w(t) + \Delta w(t),\\ &\Delta w(t) = \left(\frac{\delta\beta}{(1+\delta)(1+\gamma(t))}\frac{a(t)c^m(t)l^{1-m}(t) - w(t)l(t)}{w(t)l(t)} + \frac{\lambda}{1+\delta}\right) w(t),\\ &c(t+1) = c(t) + \Delta c(t),\\ &\Delta c(t) = \frac{\beta\gamma(t)}{1+\gamma(t)}\left(a(t)c^m(t)l^{1-m}(t) - w(t)l(t)\right).\end{aligned}$$

Output equations:

$$\begin{aligned}&y(t) = a(t)c^m(t)l^{1-m}(t),\\ &\alpha(t) = \frac{a(t)c^m(t)l^{1-m}(t)}{w(t)l(t)} - 1,\\ &r(t) = \frac{a(t)c^m(t)l^{1-m}(t) - w(t)l(t)}{c(t) + w(t)l(t)}.\end{aligned}$$

Control input:

$$\gamma(t) = h(l(t), w(t), c(t), y(t), \alpha(t), r(t)),$$

where $h(\cdot)$ is a nonlinear function that is to be designed.

Fig. 2 shows the schematic diagram of the feedback control system for the profit rate. In Park and Yang (2023), the control input $\gamma(t)$ is selected under the only constraint that it *increases* the rate of profit in the next year, i.e, $r(t+1) > r(t)$. In this study, by contrast, we design $\gamma(t)$ so as to *maximize* both the profit $p(t+1)$ and its rate $r(t+1)$ in the next year. As addressed before, further, the approximation (7) underlying the state and output equations is not valid anymore if $\gamma(t)$ is close to 0. Hence we will refine the dynamic model so that it can accurately reflect any perturbation of the closed-loop behavior caused by $\gamma(t)$.

2.2. Control input for maximizing profit and its rate

Before presenting the control input to maximize the profit and its rate, let us discuss the relationship between the composition of capital and surplus-value rate. According to Marx, an increase in the composition of capital becomes correlated with a tendential rise in the rate of surplus-value. The reason is that the rise in the productivity of labor cheapens commodities and thus the value of labor-power (the value of the means of subsistence such as foods and clothes) will decrease. Marx also states that even though both the composition of capital and surplus-value rate increase, one cannot avoid a tendential fall in the rate of profit because the rise

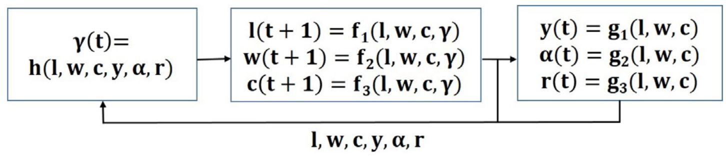


Fig. 2. Feedback control system model for the profit rate. The main problem is to design the control input $\gamma(t)$ to maximize the profit and its rate in the next year $t+1$ upon the observation of state variables $l(t), w(t), c(t)$ and output variables $y(t), \alpha(t), r(t)$.

in the composition of capital dominates the long-run behavior of the profit rate.

To proceed our discussion, the growth rates of major variables are to be identified. For readability, here we only address the derivation results and place their detailed computations in Appendix A.

Applying (4) and (5), we first obtain $\rho(c(t)/v(t))$, the growth rate of composition of capital $c(t)/v(t)$, as

$$\rho\left(\frac{c(t)}{v(t)}\right) \simeq \frac{\beta\alpha(t)}{1+\gamma(t)}\left(\frac{\gamma(t)}{c(t)/v(t)}-1\right). \tag{13}$$

The above result means that if $\gamma(t) > c(t)/v(t), \rho(c/v) > 0$, or the composition of capital has a positive growth rate. In addition, as γ increases, so does $\rho(c/v)$, which is evident since the rise in $\gamma(t)$ implies an increase in the accumulated constant capital $\Delta c(t)$ in relation to the accumulated variable capital $\Delta v(t)$ (see (4)).

$\rho(p(t))$, the growth rate of profit $p(t)$, in turn becomes

$$\rho(p(t)) \simeq \frac{\rho(a) + m\rho(c) + (1-m)\rho(l) - \Delta v/y}{1 - v/y}. \tag{14}$$

Utilizing (14), we further obtain $\rho(\alpha(t))$, the growth rate of surplus-value rate $\alpha(t)$, such that

$$\begin{aligned}\rho(\alpha(t)) &\simeq \frac{\beta\alpha}{1+\gamma}\frac{1+\alpha}{\alpha}\left(\frac{m\gamma}{c/v}+\frac{1-m}{1+\delta}-1\right)\\ &\quad+\frac{1+\alpha}{\alpha}\left(\epsilon-\frac{1-m}{1+\delta}\lambda\right) \qquad (15)\\ &\simeq \frac{1+\alpha}{\alpha}\left(m\rho\left(\frac{c}{v}\right)-\frac{(1-m)\delta}{1+\delta}\frac{\beta\alpha}{1+\gamma}\right)\\ &\quad+\frac{1+\alpha}{\alpha}\left(\epsilon-\frac{1-m}{1+\delta}\lambda\right). \qquad (16)\end{aligned}$$

It follows from (15) that when other factors are fixed, the rise in the control input $\gamma(t)$ increases the surplus-value rate at year $t+1$. Since the rise in $\gamma(t)$ implies an increase in the accumulated constant capital $\Delta c(t)$ in relation to the accumulated variable capital $\Delta v(t)$, (15) implies that the larger investment in machinery and automation in comparison with the employment of workers increases the surplus-value rate, i.e., exploitation. In view of (16), further, much as the composition of capital increases, so does the surplus-value rate.

Next, we derive $\rho(c(t)+v(t))$, the growth rate of total capital $c(t)+v(t)$, as

$$\rho(c(t)+v(t)) = \beta r(t). \tag{17}$$

The result of (17) elicits that while the growth rate of total capital is proportional to the accumulation rate β as well as the profit rate $r(t)$, it does not rely on the control input $\gamma(t)$. Finally, $\rho(r(t))$, the growth rate of profit rate $r(t)$, is derived as

$$\begin{aligned}&\rho(r(t))\\ &= \tfrac{1+\alpha}{\alpha}\left(\epsilon-\frac{1-m}{1+\delta}\lambda+\frac{m\beta\alpha v}{c}\frac{\gamma}{1+\gamma}+\left(\frac{\beta\alpha(1-m)}{1+\delta}-\frac{\beta\alpha}{1+\alpha}\right)\frac{1}{1+\gamma}\right)-\beta r,\end{aligned} \tag{18}$$

which implies that $\rho(r(t))$ depends on $\gamma(t)$.

The analyses on the growth rates so far indicate that while $\rho(r(t))$ is determined by $\gamma(t), \rho(c(t)+v(t))$ is independent of $\gamma(t)$. Therefore, it can be said that the control input $\gamma(t)$ maximizing $\rho(r(t))$ also maximizes $r(t+1)$ as well as $p(t+1)$, that is, both the profit rate and profit in the next year.

By a simple algebraic manipulation, we now obtain a proper control input $\gamma(t)$ which maximizes $\rho(r(t))$ addressed in (18). The following is the main result of this article.

Control input $\gamma(t)$ to maximize the profit and its rate at year $t+1$:

$$\gamma(t) = \begin{cases}\infty & \text{if } \dfrac{m}{c(t)/v(t)} + \dfrac{1}{1+\alpha(t)} > \dfrac{1-m}{1+\delta}\\ 0 & \text{otherwise.}\end{cases} \tag{19}$$

The detailed procedure of deriving $\gamma(t)$ is found in Appendix B. If both $c(t)/v(t)$ and $\alpha(t)$ are small enough to satisfy the condition in (19), then (18) ensures that as $\gamma(t)$ increases, so does $\rho(r(t))$, leading to an increase in the profit and its rate in the next year. Moreover, it follows from (13) and (15) that the increase in $\gamma(t)$ also leads to that in both $c(t+1)/v(t+1)$ and $\alpha(t+1)$. Such successive increases notwithstanding, there may arrive a time at which the condition in (19) is not valid. Then one must cut back $\gamma(t)$ drastically to continue to achieve the objective of maximizing the profit and its rate.

If $\gamma(t) = 0$, we have $\beta p(t) = \Delta v(t)$, namely, all the accumulated capital is used to increase variable capital, i.e., raising employment and wage. If $\gamma(t) = \infty$, on the other hand, $\Delta v(t)$ will converge to 0, so almost all the accumulated capital is expended to increase constant capital, e.g. buying machines and building factories without increasing employment and wage. In reality, however, it is rare to take an infinite $\gamma(t)$ for maximizing the profit. For example, in the U.S. economy, the maximum value of $\gamma(t)$ during 1945–2015 was 17.9 in 2000 with the average about 2.36. In general, if the control input increases over a certain value, its further increment hardly contributes to increasing the profit and its rate, as will be verified in our simulation study.

2.3. Conditions for tendential fall in profit rate

In the short run, the effort to increase the profit and its rate is likely to be successful. If $c(t)/v(t)$ and $\alpha(t)$ are so small that the condition in (19) is met, one just has to select a large $\gamma(t)$. However, the rise in the profit rate cannot endure. As $\alpha(t)$ and $c(t)/v(t)$ increase, there comes a moment when the condition in (19) is not valid anymore. Then no positive $\gamma(t)$ exists which satisfies $\rho(r(t)) > 0$. The best control policy in this case would be to set $\gamma(t)$ as small as possible, namely, maximizing the profit and its rate in the next year via minimizing their declines. This leads to $\gamma(t) = 0$.

As stated earlier, the approximation (7) is not valid with $\gamma(t) = 0$. Explicitly, $\Delta v(t)$ becomes

$$\begin{aligned}\Delta v(t) &= w(t)\Delta l(t) + l(t)\Delta w(t) + \Delta w(t)\Delta l(t)\\ &= \beta\alpha(t)v(t).\end{aligned} \tag{20}$$

Then, using (9), we obtain

$$\rho(l) = \frac{-(1+\delta+\lambda) + \sqrt{(1+\delta+\lambda)^2 + 4\delta(\beta\alpha-\lambda)}}{2\delta}.$$

The state equations of $l(t)$ and $w(t)$ should be altered accordingly. In addition, $\rho(c) = 0$ and $\rho(v) = \beta\alpha(t)$ when $\gamma(t) = 0$, and the approximations $\rho(c/v) \simeq \rho(c) - \rho(v)$ and $\rho(\alpha) = \rho(p/v) \simeq \rho(p) - \rho(v)$ do not hold either since $\rho(v)$ is not sufficiently small. With the altered dynamic model and the absence of approximations, we obtain

$$\begin{aligned}
\rho(c(t)/v(t)) &= -\frac{\beta\alpha(t)}{1+\beta\alpha(t)}, \\
\rho(\alpha(t)) &= \frac{1+\alpha(t)}{(1+\beta\alpha(t))\alpha(t)}(\epsilon + (1-m)\rho(l) - \beta\alpha(t)), \\
\rho(r(t)) &= \frac{\rho(\alpha) - \rho(c/v)\frac{c/v}{1+c/v}}{1+\rho(c/v)\frac{c/v}{1+c/v}}.
\end{aligned} \tag{21}$$

Therefore, if $\gamma(t) = 0$, the composition of capital decreases and $\rho(\alpha(t)) < 0$ follows from $\rho(r(t)) < 0$. In summary, if the condition in (19) is not met and hence $\gamma(t)$ is set to be 0 for profit maximization, all of the composition of capital, surplus-value rate, and profit rate decrease. This result encapsulates the driving mechanism that the rate of profit must fall in the short run even though capitalists pursue profit maximization.

Marx's theory of the tendential fall in the profit rate is the long-run tendency. So it is necessary to derive the conditions for the long-run decline under the repetitive rise and fall in the short run. Let $t_{i+1} = t_i + 1$ for $i = 0, 1, \ldots,$ and let $r(t_0) > r(t_1)$ and $r(t_1) < r(t_2)$, i.e., the rate of profit begins to rise at year t_2 after the fall at year t_1. We call the year t_1 a *bottom year*. Let t_{n+1} be the next bottom year, i.e., $r(t_n) > r(t_{n+1})$ and $r(t_{n+1}) < r(t_{n+2})$. For notational simplicity, denote $r(t_i) := r_i$, $\alpha(t_i) := \alpha_i, c(t_i) := c_i$, and $v(t_i) := v_i$. Then it holds that $r_{i+1} = (1 + \rho(r_i))r_i$ and

$$r_{n+1} \simeq (1 + \rho(r_1) + \cdots + \rho(r_n))r_1.$$

In order for the rate of profit to decrease in the long run, it must decrease at successive bottom years in the first, or

$$r_1 > r_{n+1},$$

which implies $\rho(r_1) + \cdots + \rho(r_n) < 0$ and is equally described by

$$\sum_{i=1}^{n}\rho(\alpha_i) < \sum_{i=1}^{n}\frac{c_i/v_i}{1+c_i/v_i}\rho(c_i/v_i).$$

Next, let $r(t_{i-1}) < r(t_i)$ and $r(t_i) > r(t_{i+1})$, i.e., the rate of profit begins to fall at t_i after the rise at t_{i-1}. We call the year t_i a *peak year*. For two successive bottom years t_1 and t_{n+1}, let t_{1+p} and t_{n+q} be two successive peak years where $0 < p < n$ and $q > 1$. Then, the second condition for the profit rate to decrease in the long run is

$$r(t_1)\left(1 + \sum_{i=1}^{p}\rho(r_i)\right) > r(t_{n+1})\left(1 + \sum_{i=1}^{q-1}\rho(r_{n+i})\right),$$

which implies that for two successive bottom years t_1 and t_{n+1}, the maximum rate of profit produced between t_1 and t_{n+1} is larger than that between t_{n+1} and the next bottom year. In summary, for the rate of profit to fall in the long run, it should decrease along both successive bottom and peak years, which is ultimately achieved by the gradual increase in the composition of capital in the long run.

2.4. Discussion

It is instructive to discuss the relationship between the proposed model and some eminent approaches regarding Marx's theory. First, Okishio (1961) proposes a famous critique of Marx's theory of the falling rate of profit by asserting that under Marx's assumptions, the profit rate must rise in the course of adopting new technologies. However, this critique is based on the supposition that the real wage is constant, which is not very relevant to real capitalist economies that involve the increase in real wages. With the extended model of Okishio's approach, on the other hand, Foley et al. (2019) state that Marx-biased technical change involves that if the wage share remains constant (i.e., the surplus-value rate is constant), the real wage rises proportionately to labor productivity and the profit rate must eventually fall. However, the wage share fluctuates in real capitalist economies.

In this regard, the proposed scheme is different from the approaches of Okishio (1961) and Foley et al. (2019). The real wage and surplus-value rate in this paper are endogenous variables changing with the control input. Without imposing the assumptions that the real wage or wage share remains fixed, the proposed nonlinear dynamic model embodies the essential mechanism that the effort of increasing the profit rate consequently leads to its gradual decrease in the long run.

Among the research avenues of traditional control theory, this study can be said to deal with a kind of the stability problem since the conditions for the profit rate to fall in the long run can be regarded as the conditions for the underlying system to be stable with respect to the value of the profit rate. In consideration of the attempt to determine the control input $\gamma(t)$ (see (19)), the present study is also analogous to the controllability problem, a key concept in control theory of which main issue is to derive conditions and design procedures for a controller taking the system toward a desired state. However, the stability and controllability problems addressed in this study differ from those of traditional control theory for the following reasons.

(i) Unlike traditional control theory, in this study the way of deriving the corresponding conditions for stabilization does not require the identification of either equilibrium points or reference trajectories of the state or output vector. Hence the preemptive analysis of the closed-loop system's behavior (i.e., the movement of profit rate under the designed control input) is not needed.
(ii) While the control law (19) is designed for the purpose of increasing the rate of profit, the main concern of this study is not to confirm that the closed-loop system exhibits the "desirable behavior," that is, both the profit and its rate increase in the long run. Rather, we focus on revealing that the designed control input causes the reverse effect, namely, not only the profit but also the rate of profit will decrease in the long run. Thus it can be said that the present study tackles a kind of contradiction problems.

3. Numerical experiments

3.1. Simulation based on the U.S. data

To validate the practicality of the proposed scheme, we first simulate the behavior of the profit rate using the parameters taken from the U.S. economy 1945–2015 data (Duménil and Lévy, 2016). In 1945, the number of workers in the U.S. was about 45 million, i.e., $l(1) = 45 \times 10^6$, the average wage per worker about 2,370 dollars, i.e., $w(1) = 2,370$, and the total capital stock about 180 billion dollars, i.e., $c(1) = 180 \times 10^9$. The average accumulation rate from 1945 to 2015 is 0.54, i.e., $\beta = 0.54$. For the production function $y(t) = a(t)c^m(t)l^{1-m}(t)$, we set $m = 0.29$ which is an average profit share for 1945–2015. Using the regression analysis of Microsoft Excel, we obtain the estimated state of technology $a(t) = 294e^{0.0355(t-1)}$ with $a(1) = 294$ and $\epsilon = 0.0355$, and a wage model $\rho(w(t)) = 0.262\rho(l(t)) + 0.0458$ with $\delta = 0.262$ and $\lambda = 0.0458$. Then, $\epsilon - \lambda(1-m)/(1+\delta) \simeq 0.01$. The control input is assigned as follows: if the condition in (19) holds true, $\gamma(t) := 10$; otherwise, $\gamma(t) := 0$.

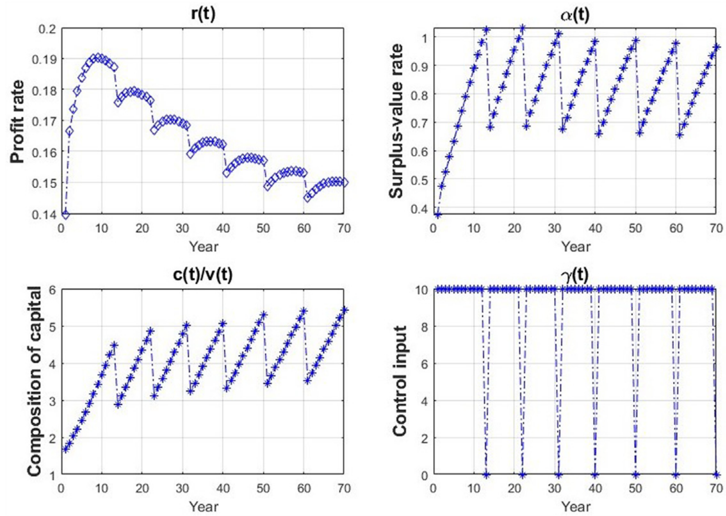


Fig. 3. Simulation results of $r(t), \alpha(t), c(t)/v(t)$, and $\gamma(t)$ based on the parameters taken from the U.S. economy 1945–2015. While the profit rate $r(t)$ is largely driven by the surplus-value rate $\alpha(t)$ in the short run, its long-run fall is primarily caused by the rise in the composition of capital $c(t)/v(t)$.

Fig. 3 illustrates the plots of $r(t), \alpha(t), c(t)/v(t)$, and $\gamma(t)$ simulated with the parameters taken from the U.S. data. It is observed that the profit rate $r(t)$ repeats the rise and fall in the short run, but it decreases in the long run, mainly due to the gradual increase in the composition of capital $c(t)/v(t)$. The condition in (19) is not met at $t = 13$, so $\gamma(13) := 0$ and the profit rate falls sharply from $r(13) = 0.187$ to $r(14) = 0.175$. Both the composition of capital and surplus-value rate increase until $t = 13$, and then drastically fall at $t = 14$, which implies that the drastic fall in the profit rate is driven by the fall in the surplus-value rate.

We also note that while the profit rate mostly increases with $\gamma(t) = 10$, it sometimes falls slightly, showing a negative growth rate, e.g., $\rho(r(11)) < 0$ and $\rho(r(12)) < 0$. Moreover, Fig. 3 demonstrates that the decrease in the profit rate in the long run is actualized by the consistent decrease along successive bottom and peak years. For example, for two successive bottom years $t = 14$ and 23, $r(14) = 0.175 > r(23) = 0.166$, and for two successive peak years $t = 18$ and 27, $r(18) = 0.179 > r(27) = 0.17$ and so on.

To validate our assertion that the proposed model accurately reflects the variable perturbations caused by the control input, let us compare the growth rates between the simulation result and analytic computation using (21). Since zero control input invokes perturbations to the variables, we select $t = 13$ and 31 at which $\gamma(13) = \gamma(31) = 0$. Table 1 shows the corresponding growth rates $\rho(r), \rho(\alpha)$, and $\rho(c/v)$ produced by the present simulation and analytic computation, respectively. It is clear that the grow rates of $r(t), \alpha(t)$, and $\rho(c/v)$ are almost identical for both cases, confirming the accuracy of the refined dynamic model and equations in (21) in the case of zero control input.

Fig. 4 shows the profit rate for various values of the control input $\gamma(t)$. One can see that as long as the condition for $\gamma(t) = \infty$ holds true in (19), the increase in the control input—in this case tenfold from $\gamma(t) = 10$ to 100 and so forth—does not change the tendential fall in the rate of profit. Put another way, the effort to maximize the profit rate cannot reverse its diminishing flow, an inherent contradiction of capitalism as asserted by Marx.

The simulation results of Figs. 3 and 4 elucidate that while the increase in both surplus-value rate and composition of capital enhances the rate of profit in the short run, the rate of profit must eventually face with a drastic fall. Furthermore, the fall in the rate of profit in the long run is caused by the progressive increase in the composition of capital and the slight decrease in the surplus-value rate in the long run. These behaviors comply with Marx's analysis that even though the rate of profit may rise due to the increase in surplus-value rate in the short run, it must decrease in the long run owing to the increase in the composition of capital.

The present simulation results also verify that the control law addressed in this study, namely, aiming at maximizing both the profit and its rate, evolves over the previous one (Park and Yang, 2023) of which objective is to merely increase the profit and its rate in the next year. To highlight this difference, we plot in Fig. 5 the profit rate $r(t)$ with respect to two different control inputs $\gamma(t) := 6$ and $\gamma(t) := 60$ for $t = 1, \ldots, 30$. It is observed that when $\gamma(t) := 6$ is applied, $r(t)$ monotonically increases during $t = 2, \ldots, 9$, which implies that $\gamma(t) := 6$ can serve as a solution in the previous control law (Park and Yang, 2023) at this time range. But we see that during $t = 2, \ldots, 9$, the value of $r(t)$ produced by the tenfold control input $\gamma(t) := 60$ is always greater than that by $\gamma(t) := 6$, that is, the greater the control input $\gamma(t)$ is, the higher the profit rate $r(t)$. This analysis confirms that the proposed control law (19) (setting a great value of $\gamma(t)$) succeeds in maximizing the profit rate in the short run.

The foregoing analysis does not lose validity with respect to the profit $p(t)$. Table 2 shows the profit $p(t)$ obtained by applying $\gamma(t) := 6$ and $\gamma(t) := 60$ separately for $t = 2, \ldots, 9$. Like the case of the profit rate, the value of $p(t)$ is enlarged at every fiscal year as $\gamma(t)$ is inflated tenfold from 6 to 60. This outcome affirms the effectiveness of the proposed control law regarding the maximization of profit (in the short run).

Table 1
Comparison between the simulation and analytical computation using (21) for the case of $t = 13$ and 31. When $\gamma(t) := 0$, analytical results by (21) are almost equal to the simulation results of Fig. 3.

Control input	$\gamma(13) := 0$			$\gamma(31) := 0$		
Growth rate	$\rho(r)$	$\rho(\alpha)$	$\rho(c/v)$	$\rho(r)$	$\rho(\alpha)$	$\rho(c/v)$
Simulation	−0.06	−0.334	−0.356	−0.054	−0.33	−0.353
Analysis by (21)	−0.056	−0.331	−0.356	−0.05	−0.33	−0.353

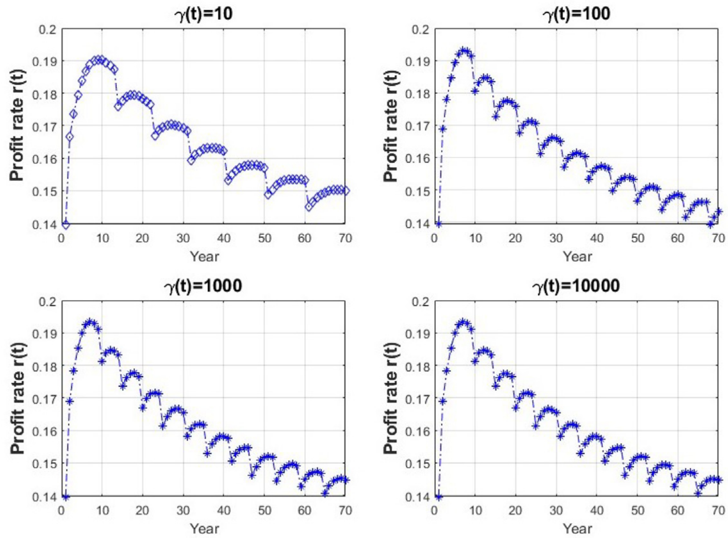


Fig. 4. Profit rate $r(t)$ for various values of the control input $\gamma(t)$. Though $\gamma(t)$ is designed for maximizing the profit and its rate in the next year, such an effort does not change the tendential fall in the rate of profit in the long run.

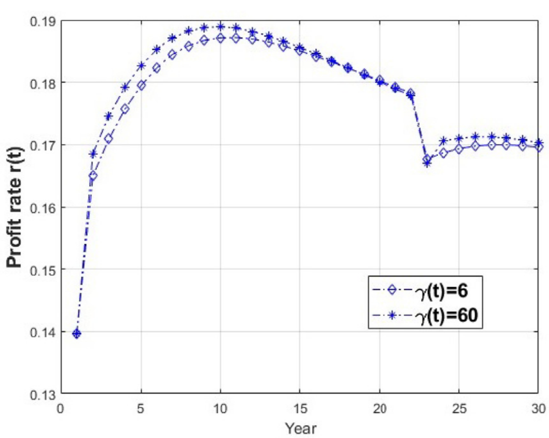


Fig. 5. Profit rate $r(t)$ with respect to two different control input $\gamma(t) := 6$ and $\gamma(t) := 60$ during $t = 1, \ldots, 30$. It is observed that when there exists some $\gamma(t)$ which increases the profit rate in the next year ($\gamma(t) := 6$ for $t = 2, \ldots, 9$), the amount of the profit rate at each year is enlarged as one inflates the value of $\gamma(t)$ ($\gamma(t) := 60$).

3.2. U.K. economy 1950–2019

Besides the foregoing numerical verification with respect to the U.S. economy data, there exist many results supporting the tendential fall in the rate of profit occurring to other countries. In fact, Basu et al. (2022) show that the world profit rate series including 170 countries display an overall declining trend from 1960 to 2019. To reinforce this assertion, we conduct another numerical experiment with respect to the U.K. economy 1950–2019 taken from the data set of the Extended Penn World Table 7.0 (EPWT7) addressed in Marquetti et al. (2021). In this simulation, the average accumulation rate is $\beta = 0.9$ with its standard deviation 0.64, the estimated wage model is $\rho(w(t)) = -1.138\rho(l(t)) + 0.078$ with $\delta = -1.138$ and $\lambda = 0.078$ (the R-squared of the regression is 0.08), the estimated state of technology is $a(t) = 43e^{0.0476(t-1)}$ with $a(1) = 43$ and $\epsilon = 0.0476$ (the R-squared of the regression is 0.95), and the average profit share is $m = 0.35$ with its standard deviation 0.024. In 1950, in particular, the number of workers in the U.K. was about 23 million, i.e., $l(1) = 23 \times 10^6$, the average wage per worker about 327 pounds, i.e., $w(1) = 327$, and the total capital stock about 22 billion pounds, i.e., $c(1) = 22 \times 10^9$.

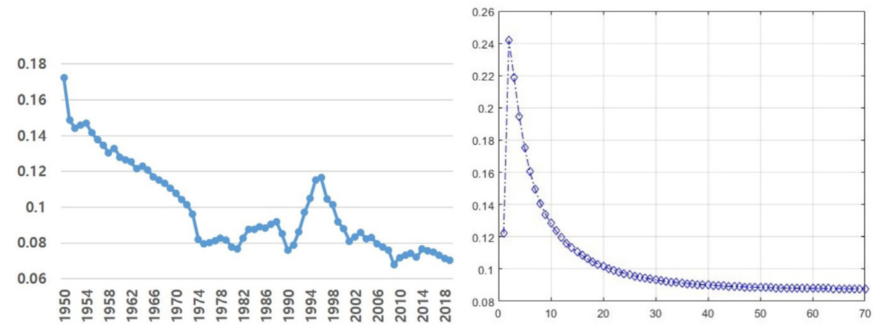
Fig. 6(a) illustrates the real profit rate of the U.K. economy from 1950 to 2019, whereas Fig. 6(b) the simulated profit rate in the proposed dynamic model using the estimated parameters of EPWT7. We see that the simulation result well follows the real behavior of the profit rate except the period of a drastic increase from 1990 to 1996. Moreover, both results are observed to converge to the steady values around 0.08.

3.3. Comparative study

Let us compare our simulation results with the empirical data derived from Duménil and Lévy (2016). Fig. 7 shows the profit rate, composition of capital, and surplus-value rate of the U.S. economy during 1949–1958 and 1960–1982, respectively, computed using the data set of Duménil and Lévy (2016). According to this figure, the real profit rate follows the movement of the surplus-value rate in the short run, which is similar to our simulation result shown in Fig. 3. Even though the short-run behavior of the composition of capital in Fig. 7 differs from that in Fig. 3, their long-run behaviors are analogous with each other in that both of them increase in the long run.

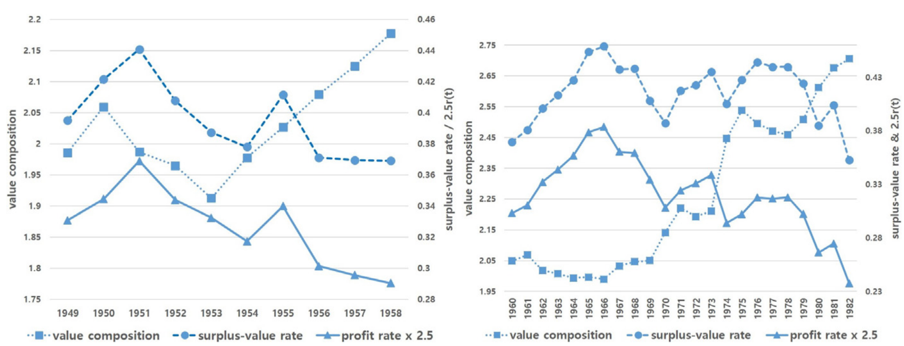
Table 2
Profit $p(t)$ (billion dollars) with respect to two different control inputs $\gamma(t) := 6$ and $\gamma(t) := 60$ during $t = 2, \ldots, 9$. It is noted that the rise in the control input $\gamma(t)$ from 6 to 60 contributes to not only the increment of the profit rate (Fig. 5), but also that of the profit, provided that there exists a control input to yield an enhancement of the profit rate in the next year.

Year	2	3	4	5	6	7	8	9
$\gamma(t) := 6$	50.84	57.35	64.4	72.03	80.27	89.19	98.84	109.28
$\gamma(t) := 60$	51.92	58.6	65.67	73.3	81.55	90.44	100.05	114.3



(a) (b)

Fig. 6. Profit rate in the U.K. during 1950–2019: (a) real data and (b) simulation result. Despite some short-run periods such as 1990–1996, the simulation result computed by the proposed model well follows the long-run behavior of the real profit rate.



(a) (b)

Fig. 7. $r(t), \alpha(t)$, and $c(t)/v(t)$ for the U.S. economy during (a) 1949–1958 and (b) 1960–1982 obtained from the data set of Duménil and Lévy (2016). During 1949–1958, while the profit rate follows the surplus-value rate in the short run, the gradual rise in composition of capital plays a major role in the fall in profit rate in the long run. Similarly, during 1960–1982, the short-run behavior of profit rate follows the surplus-value rate, and the long-run rise in composition of capital leads to the long-run decrease in profit rate.

Table 3
The long-run growth rates of profit rate, surplus-value rate, and composition of capital for two periods of 1949–1958 and 1960–1982 in the U.S. (Duménil and Lévy, 2016) and the simulation result in Fig. 3. The growth rate of profit rate in the long run is mainly determined by that of the composition of capital for both empirical data and simulation results.

	$\rho(r)$	$\rho(\alpha)$	$\rho(c/v)$
1949–1958	−0.121	−0.065	0.099
1960–1982	−0.214	−0.046	0.32
Simulation	−0.203	−0.054	0.212

In addition, Fig. 3 and Fig. 7 identically show that the surplus-value rate slightly decreases in the long run. But it cannot be ensured that this slight long-run decrease in the surplus-value rate contributes to the long-run decrease in the profit rate. To analyse this point in more detail, we list up in Table 3 the growth rates of profit rate, composition of capital, and surplus-value rate for each case of the empirical data and the simulation result. In particular, the duration $t = 13$ to $t = 70$ is taken from the simulation result in Fig. 3 wherein $r(13) = 0.187$ and $r(70) = 0.149$. As Table 3 shows, in spite of the slight long-run decrease in surplus-value rate, the long-run fall in the profit rate is largely caused by the rise in composition of capital.

In summary, our simulation results are similar to the empirical data in the sense that while the profit rate is dominated by the surplus-value rate in the short run, its long-run fall is determined by the gradual increase in the composition of capital.

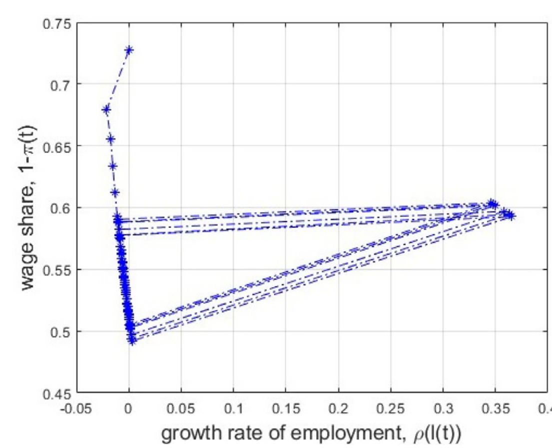


Fig. 8. Phase portrait of the wage share $1 - \pi(t)$ ($\pi(t)$ is a profit share) and growth rate of employment $\rho(l(t))$ obtained from the simulation based on the U.S. economy data. The proposed dynamic model produces limit cycles of $1 - \pi(t)$ and $\rho(l(t))$ that are similar to the Goodwin's cycles of the wage share and employment rate (Goodwin, 1967).

An interesting feature of our simulation result is that it exhibits a similar behavior to the Goodwin's cycles (Goodwin, 1967). Fig. 8 illustrates the phase portrait of the wage share $1 - \pi(t)$ and $\rho(l(t))$, the growth rate of employment, that is obtained from the simula-

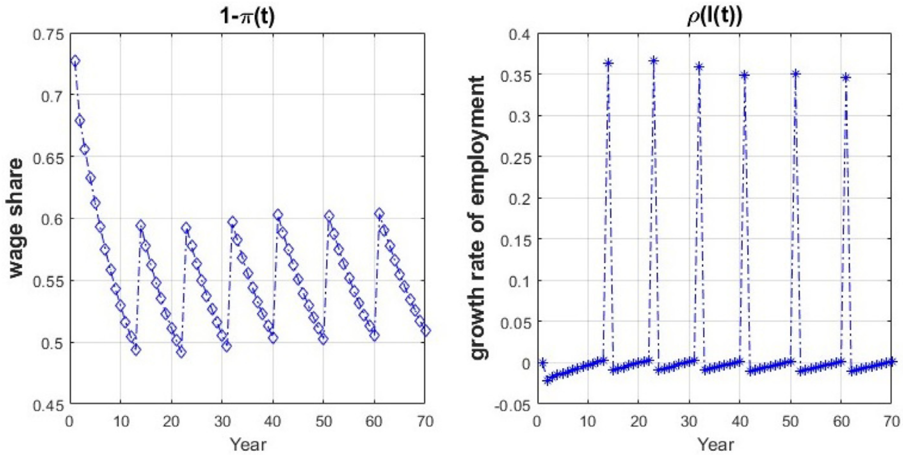


Fig. 9. Time series plot of the wage share and growth rate of employment composing the limit cycles in Fig. 8. These graphs state that if the profit share increases (i.e., the wage share decreases), the growth rate of employment increases as well.

tion using the parameters estimated with the U.S. economy data during 1945–2015. Starting from an initial condition, $1-\pi(t)$ and $\rho(l(t))$ reach a number of limit cycles bounded within a certain region, while the oscillations never cease to happen. In addition, referring to Fig. 9, time series plots of the wage share and growth rate of employment composing the limit cycles in Fig. 8, these two variables move in a counterclockwise fashion following the limit cycles. This implies that if the profit share increases (i.e., the wage share decreases), the growth rate of employment also increases. This feature is similar to the Goodwin's cycles in which the employment rate tends to increase in line with an increase in the profit share. In fact, our wage model in (9) extends the Goodwin's wage model by replacing the employment rate with the growth rate of employment. We also note that while the Goodwin's cycle is obtained using the Leontief production function, our limit cycles are derived based on the Cobb-Douglas function.

4. Conclusions

In this article, we have presented a novel feedback control approach to analyzing Marx's theory of the tendential fall in the rate of profit. The main achievement is to unravel the mechanism that any effort to maximize the profit rate in the next year, as implemented by designing a proper control input based on the nonlinear dynamic model, eventually leads to its decrease, which conforms to the inherent contradiction of capitalism. In the development of designing the control input, the relationship between the value composition of capital and the surplus-value rate has been identified. In the framework of the proposed dynamic model, the profit rate is primarily driven by the movement of the surplus-value rate in the short run due to the tendency that the rise and fall of both profit rate and surplus-value rate move in tandem. In the case of long run, on the other hand, the falling rate of profit is mainly caused by the gradual increase in the composition of capital. We have shown that this result is consistent with the empirical data in the U.S. via in-depth numerical experiments.

The result of this study is significant in that the proposed approach based on the feedback control of a dynamic model provides valuable insights into the movement of the profit rate, which cannot be acquired from the methodologies of conventional economics. In particular, whereas the most research reports on the tendential fall in the rate of profit impose the assumption that major variables such as real wage and profit share are constant, the proposed scheme elucidates the mechanism of the falling rate of profit without such assumptions. Finally, it can be said that the proposed control theoretic approach serves as a new avenue for studying the law of tendential fall in the profit rate. It is expected that various control techniques developed in the field of engineering could be applied to study the law.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

We address some detailed computations of the growth rates of major variables under the setting that the Cobb-Douglas production function $y(t)=a(t)c^m(t)l^{1-m}(t)$ is employed. In the following description, '(t)' will be omitted whenever convenient. We also remind the growth rate of variable $x(t)$ as

$$\rho(x(t))=\frac{\Delta x(t)}{x(t)}=\frac{x(t+1)-x(t)}{x(t)},$$

and the approximations $\rho(x/z)\simeq\rho(x)-\rho(z)$ and $\rho(xz)\simeq\rho(x)+\rho(z)$ with small $\rho(x)$ and $\rho(z)$ for variables x and z.

$\rho(c(t)/v(t))$, the growth rate of composition of capital $c(t)/v(t)$:

$$\begin{aligned}\rho\left(\frac{c(t)}{v(t)}\right)&\simeq\rho(c(t))-\rho(v(t))=\frac{\Delta c(t)}{c(t)}-\frac{\Delta v(t)}{v(t)}\\&=\frac{\gamma(t)\Delta v(t)}{c(t)}-\frac{\Delta v(t)}{v(t)} && (\Leftarrow(4))\\&=\frac{\beta\alpha(t)}{1+\gamma(t)}v(t)\left(\frac{\gamma(t)}{c(t)}-\frac{1}{v(t)}\right) && (\Leftarrow(5))\\&=\frac{\beta\alpha(t)}{1+\gamma(t)}\left(\frac{\gamma(t)}{c(t)/v(t)}-1\right).\end{aligned}$$

$\rho(y(t))$, the growth rate of output $y(t)$:

$$\rho(y(t)) = \rho(ac^m l^{1-m}) \simeq \rho(a) + m\rho(c) + (1-m)\rho(l).$$

$\rho(p(t))$, the growth rate of profit $p(t)$:

$$\begin{aligned}\rho(p(t)) &= \rho(y(t) - v(t)) \\ &= \frac{y(t+1) - v(t+1) - (y(t) - v(t))}{y(t) - v(t)} \\ &= \frac{(y(t+1) - y(t))/y(t) - (v(t+1) - v(t))/y(t)}{1 - v(t)/y(t)} \\ &= \frac{\rho(y) - \Delta v/y}{1 - v/y} \\ &\simeq \frac{\rho(a) + m\rho(c) + (1-m)\rho(l) - \Delta v/y}{1 - v/y}. \ (\Leftarrow \rho(y(t)))\end{aligned}$$

$\rho(\alpha(t))$, the growth rate of surplus-value rate $\alpha(t)$: We remind that $\rho(a(t)) = \epsilon$, and

$$\rho(l(t)) = \frac{1}{1+\delta}\left(\frac{\beta\alpha(t)}{1+\gamma(t)} - \lambda\right). \ (\Leftarrow (12))$$

Also, it follows from (5) that

$$\rho(v(t)) = \frac{\Delta v(t)}{v(t)} = \frac{\beta\alpha(t)}{1+\gamma(t)}$$

and compute a priori

$$\begin{aligned}&\frac{\Delta v(t)}{y(t)} = \frac{\Delta v(t)}{v(t)}\frac{v(t)}{y(t)} = \frac{\beta\alpha(t)}{1+\gamma(t)}\frac{v(t)}{y(t)}, \\ &1 - \frac{v(t)}{y(t)} = \frac{y(t) - v(t)}{y(t)} = \frac{p(t)}{y(t)} = \frac{\alpha(t)v(t)}{v(t) + \alpha(t)v(t)} = \frac{\alpha(t)}{1+\alpha(t)}.\end{aligned}$$

With this knowledge, $\rho(\alpha(t))$ is derived as

$$\begin{aligned}\rho(\alpha(t)) &\simeq \rho(p(t)) - \rho(v(t)) \\ &= \frac{\rho(a) + m\rho(c) + (1-m)\rho(l) - \Delta v/y}{1 - v/y} - \frac{\beta\alpha}{1+\gamma} \\ &= \frac{\epsilon + m\frac{\gamma\Delta v}{c} + \frac{1-m}{1+\delta}\left(\frac{\beta\alpha}{1+\gamma} - \lambda\right) - \frac{\beta\alpha}{1+\gamma}\frac{v}{y}}{1 - v/y} - \frac{\beta\alpha}{1+\gamma} \\ &= \frac{1+\alpha}{\alpha}\left(\epsilon - \frac{1-m}{1+\delta}\lambda + \frac{\beta\alpha}{1+\gamma}\left(\frac{m\gamma(t)}{c/v} + \frac{1-m}{1+\delta} - \frac{1}{1+\alpha}\right)\right) - \frac{\beta\alpha}{1+\gamma} \\ &= \frac{\beta\alpha}{1+\gamma}\frac{1+\alpha}{\alpha}\left(\frac{m\gamma}{c/v} + \frac{1-m}{1+\delta} - 1\right) + \frac{1+\alpha}{\alpha}\left(\epsilon - \frac{1-m}{1+\delta}\lambda\right),\end{aligned}$$

which is equal to (15). Further, applying the expression of $\rho(c(t)/v(t))$ derived in (i) to the above equation leads to an alternative formula

$$\begin{aligned}\rho(\alpha(t)) \simeq& \frac{1+\alpha}{\alpha}\left(m\rho\left(\frac{c}{v}\right) - \frac{(1-m)\delta}{1+\delta}\frac{\beta\alpha}{1+\gamma}\right) \\ &+ \frac{1+\alpha}{\alpha}\left(\epsilon - \frac{1-m}{1+\delta}\lambda\right),\end{aligned}$$

which is equal to (16).

$\rho(c(t) + v(t))$, the growth rate of total capital $c(t) + v(t)$:

$$\begin{aligned}\rho(c(t) + v(t)) &= \frac{c(t+1) + v(t+1) - c(t) - v(t)}{c(t) + v(t)} \\ &= \frac{\Delta c(t) + \Delta v(t)}{c(t) + v(t)} = \Delta v(t)\frac{1+\gamma(t)}{c(t) + v(t)} \\ &= \frac{\beta\alpha(t)v(t)}{c(t) + v(t)} = \beta r(t).\end{aligned}$$

$\rho(r(t))$, the growth rate of profit rate $r(t)$:

$$\begin{aligned}\rho(r(t)) &= \rho\left(\frac{p(t)}{c(t)+v(t)}\right) \\ &\simeq \rho(p(t)) - \rho(c(t) + v(t)).\end{aligned}$$

Applying $\rho(p)$ in (iv) and $\rho(c+v)$ in (17) to the above equation results in

$$\begin{aligned}&\rho(r(t)) \\ &\simeq \frac{\epsilon + m\beta\alpha\frac{v}{c}\frac{\gamma}{1+\gamma} + \frac{1-m}{1+\delta}\left(\frac{\beta\alpha}{1+\gamma} - \lambda\right) - \frac{\beta\alpha}{1+\alpha}\frac{1}{1+\gamma}}{1 - v/y} - \beta r \\ &= \frac{1+\alpha}{\alpha}\left(\epsilon - \frac{1-m}{1+\delta}\lambda + \frac{m\beta\alpha v}{c}\frac{\gamma}{1+\gamma} + \left(\frac{\beta\alpha(1-m)}{1+\delta} - \frac{\beta\alpha}{1+\alpha}\right)\frac{1}{1+\gamma}\right) - \beta r.\end{aligned}$$

Appendix B

To derive the control input $\gamma(t)$ that maximizes $\rho(r(t))$, we compute the partial derivative of $\rho(r(t))$ over $\gamma(t)$ as follows.

$$\begin{aligned}\frac{\partial\rho(r(t))}{\partial\gamma} &= \frac{1+\alpha}{\alpha}\left(\frac{m\beta\alpha v}{c}\frac{1}{(1+\gamma)^2} - \left(\frac{\beta\alpha(1-m)}{1+\delta} - \frac{\beta\alpha}{1+\alpha}\right)\frac{1}{(1+\gamma)^2}\right) \\ &= \frac{1+\alpha}{\alpha}\left(\frac{m\beta\alpha v}{c} - \left(\frac{\beta\alpha(1-m)}{1+\delta} - \frac{\beta\alpha}{1+\alpha}\right)\right)\frac{1}{(1+\gamma)^2}.\end{aligned}$$

Clearly, if

$$\frac{m\beta\alpha v}{c} - \left(\frac{\beta\alpha(1-m)}{1+\delta} - \frac{\beta\alpha}{1+\alpha}\right) > 0,$$

it holds that

$$\frac{\partial\rho(r(t))}{\partial\gamma} \geqslant 0 \ \forall\gamma,$$

and $\partial\rho(r(t))/\partial\gamma$ converges to zero as γ approaches to ∞. Hence $\rho(r(t))$ would become maximum when $\gamma(t) := \infty$. The above inequality is reduced to

$$\begin{aligned}&\frac{m\beta\alpha v}{c} - \left(\frac{\beta\alpha(1-m)}{1+\delta} - \frac{\beta\alpha}{1+\alpha}\right) > 0 \\ &\Rightarrow \frac{mv}{c} - \left(\frac{1-m}{1+\delta} - \frac{1}{1+\alpha}\right) > 0 \\ &\Rightarrow \frac{m}{c/v} + \frac{1}{1+\alpha} > \frac{1-m}{1+\delta}.\end{aligned}$$

Conversely, if

$$\frac{m}{c/v} + \frac{1}{1+\alpha} < \frac{1-m}{1+\delta},$$

$\partial\rho(r(t))/\partial\gamma < 0$ for any γ. Thus as γ decreases, $\rho(r(t))$ increases and $\gamma(t) := 0$ renders the maximum $\rho(r(t))$.

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